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The Physics of Ferromagnetism

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The Physics of Ferromagnetism

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Preface

The field of magnetism has always steadily developed and new phenomena and/or new materials have appeared. These, we must study ourselves and teach to students. The time of lecture is limited, therefore we prefer to teach basic and fundamental matters rather than new ones. During the stay in the university as the faculty to teach magnetism to students, we realized the difficulty of teaching magnetism. After lecture hours we were asked by Dr. Ascheron, the editor of Springer, to write a book related to magnetism. At first we thought that we do have not enough ability to write a book of magnetism in English. But in recent years, all people are authors and the quality of the contents of a book are judged by the readers. In addition, due to having the co-authors, and not a single author, we accepted his proposal.

This book consists of three parts; part one gives the basis of magnetism, part two discusses magnetic materials and part three, spintronics. Part one includes chapters on magnetostatics, magnetism of atoms, magnetism of solids, exchange interactions, magnetic anisotropies, magnetostrictive effects, magnetic domain, and micromagnetism. Some relations of vector analysis, group theory, and second quantization, which are summarized in the corresponding appendices, are applied in some sections. Details of mathematical processes of some equations in the text are attached in the notes at the end of the section. To help readers who are not familiar with the objects, the relations in the appendices along with equations, figures, and tables in the text are referred in many parts of the text whenever they are used.

Part two is related to magnetic materials which are roughly classified as soft and hard magnetic materials. The fundamental properties of each are described. If both magnetic properties are understood well, we can apply them to individual materials.

Part three discusses spintronics, where the basic phenomena has been known for a long time, but the research field itself is new compared with parts one and two. Chapter 1 treats the history of magnetoresistance research and classification of magnetoresistance effect. Especially, theoretical and experimental treatments of magnetoresistance effect are described in detail. Chapter 2 focuses on the tunnel magnetoresistance effect. After describing the historical background, development of MgO barrier tunnel magnetoresistance junction which triggered the giant tunnel magnetoresistance is described. Also, Heusler electrode tunnel junctions which are reported much after that is introduced. Chapter 3 is related to magnetic memory, especially the principle, development, and several issues about Magnetoresistive Random Access Memory (MRAM) research are described. In Chap. 4, accompanying technologies for the development of spintronics devices such as spin-polarization measurement and Gilbert damping constant, which are very important values in order to discuss the spin-dynamics of ferromagnets, are introduced.

This book can be used as a reference for researchers. Part one, which would also be useful for researchers, can be used as a textbook for post graduate students who have learned introductory magnetism, group theory, and advanced quantum mechanics. Being written at an advanced level we recommend undergraduate students who begin to learn magnetism to read a plainly written magnetic book first, such as Physics of Ferromagnetism (S. Chikazumi, Oxford university Press, 1997), before reading this part. Parts one and two, at least, would also be useful for engineers and technicians working in magnetic materials and devices.

Finally, we hope that this book will be much helpful for students who are beginners in the field of magnetism and also researchers who are now active in magnetic fields.

Sendai *Terunobu Miyazaki*
Changchun *Hanmin Jin*

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Part II Magnetic Materials

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Part I
Foundation of Magnetism

This part introduces fundamental magnetic phenomena and theories. It consists of eight chapters: Basis of magnetism; Magnetism of atoms; Magnetism of solids; Exchange interaction; Magnetic anisotropy; Magnetostriction effects; Magnetic domain; and Micromagnetism. Temperature is in Kelvin (K) unit unless other unit is given explicitly.

Chapter 1
Basis of Magnetism

This chapter reviews mostly the basic macroscopic magnetism. It contains the sections of Basic magnetic laws and magnetic quantities; Magnetic Coulomb's law, static magnetic field, and magnetic circuit; Zeeman energy, magnetization energy, and magnetostatic energy; Thermodynamics for magnetic media; and Hamiltonian of an electric charged particle in static electric and magnetic fields; and Appendices 1 to 3.

1.1 Basic Magnetic Laws and Magnetic Quantities

1.1.1 Basic Laws of Magnetic Forces, Magnetic Induction Vector, and Magnetic Moment

A particle with electric charge q moving in a magnetic field with velocity $\vec{v}$ experiences the Lorentz force (1892)

$$\vec{f} = q\vec{v} \times \vec{B}. \tag{1.1}$$

This relation, called Lorentz force law, also defines the quantity of magnetic induction vector (often tersely called magnetic induction) $\vec{B}$ (T) characterizing the magnetic field. For example, the creatures on the Earth are protected from the solar wind, a stream of energetic particles of electrons, positrons, He^{+1} ions, etc by the magnetosphere, the Earth's magnetic field space extending tens of kilometers outward, which deflects away the charged particles by the Lorentz force. $B \sim 3 \times 10^{-5}$ T on the surface of the Earth.

A particle in an inhomogeneous magnetic field $\vec{B}$ in vacuum experiences the force

$$\vec{f} = \left(\vec{p}_M \cdot \nabla\right)\vec{B}. \tag{1.2}$$

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This relation also defines the quantity of magnetic moment $\vec{p}_M(\text{A}\cdot\text{m}^2)$ characterizing the magnetic property of the particle. The magnitude of the quantity per unit mass is called specific magnetization. The equipment of magnetic balance measures the value of magnetic moment directly by using (1.2) through measuring the force. Another application of (1.2) is magnetic separation. A gradient magnetic field produced by a permanent magnet or electromagnet separates large specific magnetization particles from small specific magnetization particles by attracting the former but leaving the latter to drop away freely.

1.1.2 Vectors of Magnetization, Magnetic polarization and Magnetic field, and Magnetic Polarization Moment

Define the vector sum of the magnetic moments per unit volume

$$\vec{M} = \lim_{v\to 0}\frac{\sum_v \vec{p}_M}{v}(\text{A/m}) \qquad (v\text{: volume}) \tag{1.3}$$

magnetization vector (often tersely called magnetization).

The magnetic polarization moment $\vec{p}_J$ and magnetic polarization vector (often tersely called magnetic polarization) $\vec{J}$ is defined by the relation

$$\vec{p}_J = \mu_0\vec{p}_M \qquad (\text{Wb}\cdot\text{m}) \tag{1.4}$$

and

$$\vec{J} = \mu_0\vec{M}(\text{T}), \tag{1.5}$$

respectively. Here $\mu_0(\text{N/A}^2)$ is the magnetic constant (also called vacuum permeability). $\vec{p}_J$ and $\vec{p}_M$, and $\vec{J}$ and $\vec{M}$, respectively, describe the same magnetic quantity but with different units. Appropriate unit quantities are used in different cases to make the formulations succinct.

Magnetic field vector (often tersely called magnetic field) $\vec{H}$ is defined by the relation

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}(\text{A/m}). \qquad \left(\vec{B} = \mu_0\vec{H} + \vec{J}(\text{T})\right) \tag{1.6}$$

In vacuum, $\vec{J} = 0$, so $\vec{H} = \vec{B}/\mu_0$ and $\vec{B}$ describes the same magnetic field using different units.

1.1.3 Maxwell Equations

The Maxwell equations (1861–1862) are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \left(\vec{E}\text{: electric field vector, } t\text{: time}\right) \tag{1.7}$$

$$\nabla \cdot \vec{D} = \rho, \quad (\rho\text{: free electric charge density}) \tag{1.8}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \quad \left(\vec{j}\text{: conducting electric current density}\right) \tag{1.9}$$

$$\nabla \cdot \vec{B} = 0, \tag{1.10}$$

where $\vec{D} = \varepsilon\varepsilon_0 \vec{E}$ is electric displacement vector, ε relative permittivity (often tersely called permittivity), ε_0 electric constant (also called vacuum permittivity), and $\varepsilon\varepsilon_0$ permittivity.

1.1.4 Magnetic Vector Potential

Equation (1.10) shows that $\vec{B}$ is continuous everywhere, so it is a curl of a vector potential $\vec{A}$ (a3.14):

$$\vec{B} = \nabla \times \vec{A}. \tag{1.11}$$

$\vec{A}$ is called magnetic vector potential. The integration of it along a closed circuit l is the magnetic flux through the circuit surface:

$$\oint_l \vec{A} \cdot \mathrm{d}\vec{l} = \int \vec{B} \cdot \mathrm{d}\vec{S} \text{ (Wb).} \qquad \text{(a3.27)} \tag{1.12}$$

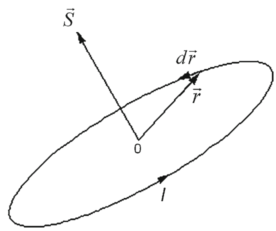
The linear integration is along the right-hand direction about the circuit surface vector $\vec{S}$ direction (Fig. 1.1). Since

$$\nabla \times \left(\vec{A} + \nabla\Psi\right) = \nabla \times \vec{A} = \vec{B} \qquad \text{(a3.10)} \tag{1.13}$$

for arbitrary function $\Psi\left(\vec{r}\right)$, both $\vec{A} + \nabla\Psi$ and $\vec{A}$ depict the same $\vec{B}$, i.e., $\vec{A}$ can be represented by many different expressions. The expression will be restricted by the conditions that it does not include a gradient of any function and

$$\nabla \cdot \vec{A} = 0. \tag{1.14}$$

Fig. 1.1 Closed current coil



1.1.5 Magnetic Moment

In a static magnetic field in vacuum where no current flows, the relations

$$\frac{\partial B_\alpha}{\partial \beta} = \frac{\partial B_\beta}{\partial \alpha} \qquad (\alpha, \beta = x, y, z, \nabla \times \vec{B} = \mu_0 \nabla \times \vec{H} = 0) \tag{1.15}$$

hold, and (1.2) can be reformulated to

$$\vec{f} = \sum_{\alpha,\beta}^{x,y,z} p_{M\beta} \frac{\partial B_\alpha}{\partial \beta} \vec{e}_\alpha = \sum_{\alpha,\beta}^{x,y,z} p_{M\beta} \frac{\partial B_\beta}{\partial \alpha} \vec{e}_\alpha = \sum_{\alpha}^{x,y,z} \frac{\vec{p}_M \cdot \partial \vec{B}}{\partial \alpha} \vec{e}_\alpha = \frac{\vec{p}_J \cdot \partial \vec{H}}{\partial \vec{r}}, \tag{1.16}$$

where $\vec{e}_\alpha$ represents the unit vector in the α direction.

Consider the magnetic polarization moment $\vec{p}_J$ and the magnetic field as a system. If $\vec{p}_J$ is shifted by $\mathrm{d}\vec{r}$ under the action of the outside force $-\vec{f}$ counterbalancing the internal force of (1.16), the energy of the system E increases by

$$\mathrm{d}E = -\vec{f} \cdot \mathrm{d}\vec{r} = -\vec{p}_J \cdot \mathrm{d}\vec{H}. \tag{1.17}$$

Therefore, $\vec{p}_J$ is related with E by

$$\vec{p}_J = -\frac{\partial E}{\partial \vec{H}}. \tag{1.18}$$

1.1.6 Magnetic Moment of Current Coil

One example of magnetic moment is the magnetic moment of a small current coil. Let

$$\vec{S} = \frac{1}{2} \oint_S \vec{r} \times \mathrm{d}\vec{r} \tag{1.19}$$

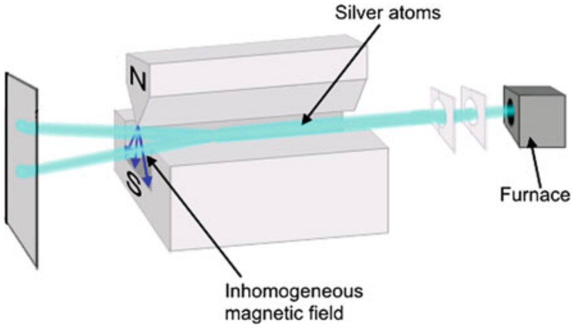


Fig. 1.2 Stern-Gerlach experiment [1]

denote the surface vector of the coil (Fig. 1.1) and I the current flowing in the right-hand direction about the $\vec{S}$ direction. The Lorentz force sensed by the current coil in an inhomogeneous magnetic field is

$$\vec{f} = \frac{I\vec{S}\cdot\partial\vec{B}}{\partial\vec{r}}. \qquad \text{(Note 1 of this section)} \tag{1.20}$$

Comparison of this relation with (1.16) reveals that the current coil has a magnetic moment of

$$\vec{p}_M = I\vec{S}. \tag{1.21}$$

1.1.7 Magnetic Moment of Electron Spin

Another example of magnetic moment is the magnetic moment of electron spin. Consider the Stern-Gerlach experiment (1922) in which a thin vapor beam of H, Li, Ag, or another atom of the s-block of the periodic table (group 1 and group 2), formed in a furnace and passing through successively two small apertures separated by certain distance, travels through an inhomogeneous magnetic field and reaches the detector. The magnetic field and its gradient directions are up and down perpendicular to the beam (Fig. 1.2). When the magnetic field is absent the beam does not deflect. If the magnetic field is applied, the beam splits up and down symmetrically under the action of the magnetic force of (1.16). The s-block atom has an s valence electron, and the atomic orbital magnetic moment is zero (Chap. 2). The experiments conclude that an electron has an intrinsic magnetic moment called electron spin magnetic moment, the projects of it in the field direction has only two components of same magnitude and different signs.

1.1.8 Magnetic Field Strength, Magnetic Induction, Magnetization, Permeability, and Susceptibility

The magnitude H of $\vec{H}$ is called magnetic field strength (often tersely called magnetic field), and the projection of $\vec{B}$, $\vec{J}$, and $\vec{M}$ in the direction of $\vec{H}$ B, J, and M is called magnetic induction, magnetic polarization, and magnetization, respectively.

$$\mu\mu_0 = \frac{B}{H} = \frac{J + \mu_0 H}{H} = (\chi + 1)\mu_0 \qquad (\mathrm{N/A^2}) \tag{1.22}$$

is called permeability, $\mu = \chi + 1$ is called relative permeability (often tersely called permeability), $\chi\mu_0 = J/H$ is called susceptibility, and $\chi = M/H$ is called relative susceptibility (often tersely called susceptibility).

Note 1

Exploiting the relations of

$$\vec{B}(\vec{r}) \approx \vec{B}(0) + \sum_{\alpha}^{x,y,z} \left.\frac{\partial \vec{B}}{\partial \alpha}\right|_0 \alpha, \tag{1.23}$$

$$\frac{\partial \vec{B}}{\partial \alpha} \equiv \left.\frac{\partial \vec{B}}{\partial \alpha}\right|_0 = \frac{\partial \vec{B}(\vec{r})}{\partial \alpha}, \tag{1.24}$$

and

$$\frac{\partial B_z}{\partial z} = -\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \qquad (1.10) \tag{1.25}$$

the component x of (1.20), for instance, is proved as

$$\begin{aligned} X &\equiv \frac{1}{I}\left(\vec{f} - \frac{I\vec{S}\cdot\partial\vec{B}}{\partial\vec{r}}\right)_x = \left[\oint d\vec{r} \times \vec{B}(\vec{r}) - \frac{1}{2}\oint \frac{(\vec{r} \times d\vec{r})\cdot\partial\vec{B}}{\partial\vec{r}}\right]_x \\ &= \oint \left[dy\left(\frac{\partial B_z}{\partial x}x + \frac{\partial B_z}{\partial y}y + \frac{\partial B_z}{\partial z}z\right) - dz\left(\frac{\partial B_y}{\partial x}x + \frac{\partial B_y}{\partial y}y + \frac{\partial B_y}{\partial z}z\right)\right] \\ &\quad - \frac{1}{2}\oint\left[(ydz - zdy)\frac{\partial B_x}{\partial x} + (zdx - xdz)\frac{\partial B_y}{\partial x} + (xdy - ydx)\frac{\partial B_z}{\partial x}\right] \\ &= \frac{1}{2}\frac{\partial B_z}{\partial x}\oint d(xy) + \frac{1}{2}\frac{\partial B_z}{\partial y}\oint\left(dy^2 - dz^2\right) - \frac{1}{2}\frac{\partial B_y}{\partial x}\oint d(xz) \\ &\quad - \frac{1}{2}\frac{\partial B_x}{\partial x}\oint d(yz) - \frac{\partial B_y}{\partial y}\oint d(yz) = 0. \end{aligned} \tag{1.26}$$

1.2 Magnetic Coulomb's law, Static Magnetic Field, and Magnetic Circuit

Consider a static magnetic system.

1.2.1 Magnetic Charge and Magnetic Coulomb's Law

Define

$$\rho_\mathrm{m} = -\nabla\cdot\vec{J} \tag{1.27}$$

as volume magnetic charge density. Inserting (1.6) and (1.27) into (1.10) we get

$$\nabla\cdot\vec{H} = \frac{\rho_\mathrm{m}}{\mu_0}. \tag{1.28}$$

$\vec{H}$ is the sum of the magnetic field vectors produced by electric currents and magnetic media $\vec{H}_I$ and $\vec{H}_\mathrm{m}$:

$$\vec{H} = \vec{H}_I + \vec{H}_\mathrm{m}. \tag{1.29}$$

The Maxwell equations for $\vec{H}_I$ are

$$\nabla\times\vec{H}_I = \vec{j}, \qquad \nabla\cdot\vec{H}_I = 0, \tag{1.30}$$

and the relations of $\vec{H}_I$ with the magnetic induction $\vec{B}_I$ and magnetic vector potential $\vec{A}_I$ both produced by electric current are

$$\vec{B}_I = \mu_0\vec{H}_I = \nabla\times\vec{A}_I. \tag{1.31}$$

The Maxwell equations for $\vec{H}_\mathrm{m}$ are completely the same in form with those of static electric field $\vec{E}$ in vacuum, that is

$$\nabla\times\vec{H}_\mathrm{m} = 0, \qquad \nabla\cdot\vec{H}_\mathrm{m} = \frac{\rho_\mathrm{m}}{\mu_0}, \tag{1.32}$$

$$\nabla\times\vec{E} = 0, \qquad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \tag{1.33}$$

in which $\vec{H}_\mathrm{m}$, ρ_m, and μ_0 corresponds to $\vec{E}$, ρ and ε_0, respectively. The left relations of (1.32) and (1.33) show that $\vec{H}_\mathrm{m}$ as well as $\vec{E}$ are gradients of scalar potentials (a3.13),

$$\vec{H}_\mathrm{m} = -\nabla V_\mathrm{m}, \tag{1.34}$$

$$\vec{E} = -\nabla V. \qquad (V\text{: electric potential}) \tag{1.35}$$

V_m is called magnetic scalar potential or tersely magnetic potential. The symmetrical relations of (1.32) with (1.33) and (1.34) with (1.35) reveal that the expressions of $\vec{H}_m$ and V_m produced by a magnetic point charge q_m at a position $\vec{r}$ from the charge are the same in form with those of $\vec{E}$ and V produced by an electric point charge q, that is

$$\vec{H}_m = \frac{q_m}{4\pi\mu_0 r^3}\vec{r}, \qquad \vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}, \tag{1.36}$$

$$V_m = \frac{q_m}{4\pi\mu_0 r}, \qquad V = \frac{q}{4\pi\varepsilon_0 r}. \tag{1.37}$$

In vacuum, the force experienced by a magnetic body in an applied magnetic field $\vec{H}$ is

$$\vec{f} = \int \left(\vec{J}\cdot\nabla\right)\vec{H}\,\mathrm{d}v = \oint \vec{H}\sigma_m \mathrm{d}S + \int \vec{H}\rho_m \mathrm{d}v, \qquad \text{(Note 1 of this section)} \tag{1.38}$$

where

$$\sigma_m \equiv \vec{J}\cdot\frac{\mathrm{d}\vec{S}}{\mathrm{d}S} \tag{1.39}$$

is the surface magnetic charge density on the surface of the magnetic body and $\mathrm{d}\vec{S}$ is the differential surface vector normal to the surface and directed outward. Equation (1.38) shows that a magnetic point charge $q_m (= \sigma_m \mathrm{d}S$ or $= \rho_m \mathrm{d}v)$ senses a force

$$\vec{f} = q_m\vec{H}. \tag{1.40}$$

In the case where $\vec{H}$ is the magnetic field produced by a magnetic point charge q'_m (1.36) apart r from q_m, (1.40) becomes the magnetic Coulomb's law

$$\vec{f} = \frac{q_m q'_m \vec{r}}{4\pi\mu_0 r^3}. \tag{1.41}$$

This law had been found experimentally in 1785 and has ever been the fundamental law of magnetism until up to the end of the following century. Correspondingly, the magnetic charge and magnetic field vector had been the fundamental magnetic quantities. According to modern magnetism, the fundamental laws are (1.1) and (1.2) and Maxwell equations (1.7)–(1.10). Correspondingly, the fundamental magnetic quantities are magnetic induction vector and magnetic moment.

The above arguments show that the behaviors of q_m and $\vec{H}_m$ are completely the same with those of q and $\vec{E}$. The magnetic Coulomb's law is very useful as is demonstrated below.

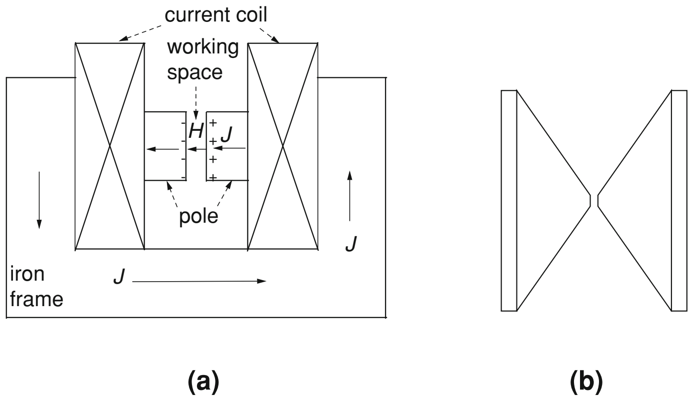


Fig. 1.3 **a** Flat pole surface electromagnet; **b** Cone-shaped pole

1.2.2 Examples of Application of Magnetic Coulomb's Law

1.2.2.1 Magnetic Field of Electromagnet

Consider an electromagnet (Fig. 1.3).

The pure Fe metal core is magnetized by the current coils wrapped around the poles. The magnetic field in the work space between the pole faces is mainly produced by the positive and negative magnetic charges on the vis-a-vis pole faces. The field produced by the current coil is much smaller. Usually Fe-35%Co alloy, which has the largest value of saturation magnetic polarization of $J_s = 2.46$ T at room temperature (R.T.), is used as the pole material. Approximately 2%V is added in the alloy to improve the mechanical strength. If a flat pole (Fig. 1.3a) is used and the gap between the pole faces is narrow, the magnetic induction in the space is $\mu_0 H = J$ T and its upper limit is $J_s = 2.46$ T. The attraction between the two pole surfaces can reach as large as $J_s^2/2\mu_0 = 2.4 \times 10^6$ N/m^2. If the pole is made appropriately cone shaped with small flat top surface (Fig. 1.3b), the upper limit of the field at the center of the space can reach as large as ~4 T but at the expense of much smaller work space.

1.2.2.2 Magnetic Moment of Magnetic Dipole

Consider a tiny columnar magnetic bar of length l and cross-section area S with the magnetic polarization vector $\vec{J}$ in the length direction $\vec{l}$. The magnetic charges $q_m = JS$ and $-q_m$ appear on the two base surfaces which constitute a magnetic dipole in the same way as an electric dipole constituted by a pair of positive and negative electric charges $\pm q$. The magnetic dipole in an inhomogeneous field senses a force of

$$\vec{f} = q_m \left[\vec{H}\left(\vec{r}_{q_m}\right) - \vec{H}\left(\vec{r}_{-q_m}\right) \right] = \left(q_m \vec{l} \cdot \nabla \right) \vec{H}. \tag{1.42}$$

Comparison of (1.42) with (1.2) reveals that magnetic dipole $q_\mathrm{m}\vec{l}$ has a magnetic polarization moment of

$$\vec{p}_J = q_\mathrm{m}\vec{l}. \tag{1.43}$$

1.2.2.3 Torque Sensed by a Magnetic Moment in a Magnetic Field

The magnetic dipole in the field $\vec{H}$ is under action of torque

$$\vec{L} = \vec{l} \times q_\mathrm{m}\vec{H} = \vec{p}_J \times \vec{H}. \tag{1.44}$$

The last of the above equation is the general expression of the torque sensed by a magnetic moment in a magnetic field.

1.2.2.4 Magnetic Field Produced by a Magnetic Moment

V_m and $\vec{H}_\mathrm{m}$ produced by a magnetic moment of magnetic dipole at $\vec{r}(r \gg l)$ from the dipole are

$$V_\mathrm{m} = \frac{q_\mathrm{m}}{4\pi\mu_0 r_{q_\mathrm{m}}} - \frac{q_\mathrm{m}}{4\pi\mu_0 r_{-q_\mathrm{m}}} = \frac{\vec{p}_M\cdot\vec{r}}{4\pi r^3},$$
$$(r_{q_\mathrm{m}}\text{: the distance from magnetic charge } q_\mathrm{m}) \tag{1.45}$$

$$\vec{H}_\mathrm{m} = -\nabla V_\mathrm{m} = \frac{1}{4\pi}\left[\frac{3\left(\vec{p}_M\cdot\vec{r}\right)\vec{r}}{r^5} - \frac{\vec{p}_M}{r^3}\right]. \tag{1.46}$$

The last expressions of (1.45) and (1.46) are the general expressions of V_m and $\vec{H}_\mathrm{m}$ produced by a magnetic moment. Since

$$\nabla\cdot\vec{H}_\mathrm{m}\left(\vec{r}\right) = 0, \qquad ((1.32),\ \rho_\mathrm{m} = 0) \tag{1.47}$$

it holds

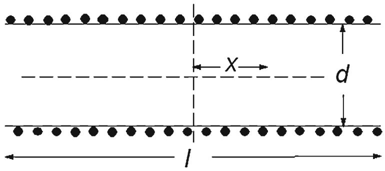
$$\mu_0\vec{H}_\mathrm{m} = \nabla \times \vec{A}_\mathrm{m}\left(\vec{r}\right). \tag{1.48}$$

$\vec{A}_\mathrm{m}\left(\vec{r}\right)$ produced by $\vec{p}_J$ is

$$\vec{A}_\mathrm{m}\left(\vec{r}\right) = \frac{\vec{p}_J \times \vec{r}}{4\pi r^3} = \frac{1}{4\pi}\nabla \times \frac{\vec{p}_J}{r}. \qquad \text{(Note 2 of this section)} \tag{1.49}$$

For a uniformly magnetized sphere, (1.45) through (1.49) holds strictly at any position outside the magnet (Note 3 of this section).

Fig. 1.4 Solenoid



1.2.3 Magnetic Field Produced by Electric Current

$\vec{A}_I$ produced by an electric current distribution of current density $\vec{j}$ is (Note 4 of this section)

$$\vec{A}_I = \frac{\mu_0}{4\pi}\int \frac{\vec{j}}{r}\mathrm{d}v. \quad (r\text{: distance between the observation point and } \vec{j}\mathrm{d}v) \tag{1.50}$$

$\vec{H}_I$ produced by current I flowing in the circuit l is

$$\begin{aligned}\vec{H}_I &= \frac{1}{\mu_0}\nabla\times\vec{A}_I = \frac{I}{4\pi}\oint\nabla\times\frac{\mathrm{d}\vec{l}}{r} = \frac{I}{4\pi}\oint(\nabla\frac{1}{r})\times\mathrm{d}\vec{l} \\ &= \frac{I}{4\pi}\oint\frac{\mathrm{d}\vec{l}\times\vec{r}}{r^3} \quad (r\text{: distance between the observation point and } \mathrm{d}\vec{l}\text{, (a3.18))}\end{aligned} \tag{1.51}$$

which is just the Biot-Savart law.

A solenoid is used to produce a field within it. The magnetic field on the axis in a solenoid of length l, diameter d, number of coil per unit length n, and current I (Fig. 1.4) is in the axis direction, and the strength at distant x from the center of the solenoid is

$$H_I = n\frac{I}{2}\left[\frac{l+2x}{\sqrt{d^2+(l+2x)^2}} + \frac{l-2x}{\sqrt{d^2+(l-2x)^2}}\right], \tag{1.52}$$

and if $l \gg d$, the magnetic field is very uniform in a wide range of space near the center of the solenoid with the value

$$H_I = nI. \tag{1.53}$$

The upper limit of the field is restricted by the increase of the temperature of the solenoid caused by the Joule heat, and in usual $\mu_0 H$ is smaller than $\sim$0.2 T.

A small number of high magnetic field laboratories around the world can produce as high as 30 T static fields by tremendously powerful cooling of the solenoid. A superconductor solenoid (also called superconducting magnet) has no electric

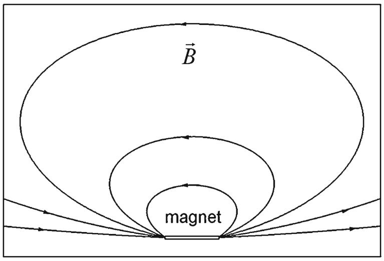


Fig. 1.5 Magnetic induction lines in the space produced by a magnetic bar

resistance below the critical current of superconductivity. The current can be very large which can produce as high as 30 T static magnetic field.

As large as over 100 T pulse magnetic field can be produced in a coil by instantly discharging the electric charge stored in a very large capacitance. The upper limit of the field is restricted by mechanical destruction of the coil under the strong repulsing forces between the current segments.

1.2.4 Volume Integral of Scalar Products of $\vec{H}_{\mathrm{m}}$ with $\vec{H}_I$, $\vec{B}_{\mathrm{m}}$, and $\vec{B}$

If static magnetic media and electric currents are distributed in a finite space, it holds

$$\int_{\infty} \vec{H}_I \cdot \vec{H}_{\mathrm{m}} \mathrm{d}v = \int_{\infty} \vec{B}_{\mathrm{m}} \cdot \vec{H}_{\mathrm{m}} \mathrm{d}v = \int_{\infty} \vec{B} \cdot \vec{H}_{\mathrm{m}} \mathrm{d}v = 0. \quad \text{(Note 5 of this section)} \tag{1.54}$$

1.2.5 Magnetic Circuit

Consider the space distribution of $\vec{B}$ produced by a thin uniformly magnetized bar in the axis direction (Fig. 1.5). The total magnetic flux emitted from the bar into the space is

$$\Phi = \int \mathrm{d}\Phi = \int_S \vec{B} \cdot \mathrm{d}\vec{S}. \quad \left(\int_S \text{is over the central section of the bar} \right) \tag{1.55}$$

A locus of $\vec{B}$ forms a closed magnetic induction line (Fig. 1.5). A magnetic flux tube is formed with the induction lines covering the tube surface surrounding a small magnetic flux $\Delta\Phi = B\mathrm{d}S$ passing through the section. A pair of magnetic charges is formed on the two base surfaces of the bar and the integration of H in the space along the induction line l in a flux tube between the two bases is

$$\int_{+\text{base}}^{-\text{base}} H\mathrm{d}l = \Delta\Phi \int_{+\text{base}}^{-\text{base}} \frac{\mathrm{d}l}{\mu\mu_0\mathrm{d}S} = \Delta\Phi r_\mathrm{m} = \Delta V_\mathrm{m}. \tag{1.56}$$

$\frac{\mathrm{d}l}{\mu\mu_0\mathrm{d}S}$ is called the magnetic resistance of the segment and $r_\mathrm{m} = \int_{+\text{base}}^{-\text{base}} \frac{\mathrm{d}l}{\mu\mu_0\mathrm{d}S}$ the magnetic resistance of the tube between the two bases. The values of the potential difference ΔV_m are the same for all parallel connected tubes. These parallel connected magnetic circuits completely correspond to the parallel electric circuits, (1.55) corresponds to the electric counterpart of Kirchhoff's current law that the total current of a parallel connected circuit system is the sum of the currents of the circuits, and (1.56) corresponds to the Ohm's law. Here ΔV_m, $\Delta\Phi$, and r_m correspond to the electric potential difference, a current flowing in a parallel connected circuit and the electric resistance of the circuit, respectively. Equation (1.56) shows that under a given ΔV_m, smaller the magnetic resistance of a magnetic circuit larger the magnetic flux in the circuit. One application of the above concept is magnetic shielding. The work space is covered by a wall made up by a large $\mu(\sim 10^5)$ magnetic material. The magnetic flux of the field of the Earth ($\mu_0 H \sim 3 \times 10^{-5}$ T) and the other field near the work space are drawn into the wall thus decreasing the field in the work space by several orders. In this way, for instance, an extremely low magnetic field space is constructed for the research of human body magnetic field of $10^{-13} \sim 10^{-8}$ T [2].

Note 1

The α component of the force is

$$\begin{aligned} f_\alpha &= \int \vec{J}\cdot\nabla H_\alpha \mathrm{d}v && (1.2) \\ &= \int [\nabla\cdot(H_\alpha \vec{J}) - H_\alpha \nabla\cdot\vec{J}]\mathrm{d}v && (\text{a}3.16) \\ &= \oint H_\alpha \sigma_\mathrm{m} \mathrm{d}S + \int H_\alpha \rho_\mathrm{m} \mathrm{d}v. && ((\text{a}3.25), (1.39), (1.27)) \end{aligned} \tag{1.57}$$

Note 2

$$\begin{aligned} \mu_0 \vec{H}_\mathrm{m} &= \nabla \times \vec{A}_\mathrm{m}(\vec{r}) \\ &= -\mu_0 \nabla V_\mathrm{m}(\vec{r}) = -\nabla\left(\frac{\vec{p}_J\cdot\vec{r}}{4\pi r^3}\right) && ((1.48), (1.46), (1.45)) \\ &= -\vec{p}_J \times \left(\nabla \times \frac{\vec{r}}{4\pi r^3}\right) - (\vec{p}_J\cdot\nabla)\frac{\vec{r}}{4\pi r^3} && (\text{a}3.15) \end{aligned}$$

$$
\begin{aligned}
&= \left(\nabla\cdot\frac{\vec{r}}{4\pi r^3}\right)\vec{p}_J - (\vec{p}_J\cdot\nabla)\frac{\vec{r}}{4\pi r^3} \qquad \left(\nabla\times\frac{\vec{r}}{r^3} = \nabla\cdot\frac{\vec{r}}{r^3} = 0 \quad (r\neq 0)\right)\\
&= \nabla\times\left(\vec{p}_J\times\frac{\vec{r}}{4\pi r^3}\right)\\
&= \nabla\times\left[\left(\nabla\frac{1}{4\pi r}\right)\times\vec{p}_J\right] \qquad \text{(a3.19)}\\
&= \nabla\times\left(\nabla\times\frac{\vec{p}_J}{4\pi r}\right). \qquad \text{(a3.18)}
\end{aligned} \tag{1.58}
$$

Note 3

V_{m} satisfies the Laplace equation axially symmetrical around the magnetization direction:

$$
\nabla^2 V_{\mathrm{m}} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial V_{\mathrm{m}}}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_{\mathrm{m}}}{\partial\theta}\right)\right] = 0, \qquad \text{(a3.23)} \tag{1.59}
$$

where θ is the angle between $\vec{r}$ and the magnetization direction. Inserting

$$
V_{\mathrm{m}}(r,\theta) = R(r)\Theta(\theta) \tag{1.60}
$$

into (1.59) we get

$$
\frac{1}{R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right] = -\frac{1}{\Theta(\theta)\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right]. \tag{1.61}
$$

The left of the equation is the function of r, and the right is the function of θ, so they should equal to a constant λ. A solution of (1.61) is $\lambda = l(l+1)$ ($l = 0$ or a positive integer),

$$
R_l(r) = a_l r^l + \frac{b_l}{r^{l+1}}, \qquad (a_l, b_l\text{: constants}) \tag{1.62}
$$

$$
\Theta_l(\theta) = P_l(\cos\theta) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}(\cos\theta)^l}(\cos^2\theta - 1)^l, \tag{1.63}
$$

where $P_l(\cos\theta)$ is the Legendre polynomial. The common solution of (1.59) is

$$
V_{\mathrm{m}}(r,\theta) = \sum_{l}^{0,1,2\cdots}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{1.64}
$$

Applying the boundary conditions of

$$V_{\mathrm{m}}(r \gg r_0, \theta) = \frac{\vec{p}_M \cdot \vec{r}}{4\pi r^3} = \frac{p_M}{4\pi r^2} P_1(\cos\theta), \quad (r_0\text{: radius of the magnet}) \tag{1.65}$$

$$V_{\mathrm{m}}\left(\vec{r}_{0-}\right) = \frac{1}{3} M r_0 \cos\theta = V_{\mathrm{m}}\left(\vec{r}_{0+}\right) = \sum_{l}^{0,1,2\cdots} \left(a_l r_0^l + \frac{b_l}{r_0^{l+1}} \right) P_l(\cos\theta), \tag{1.66}$$

where $V_{\mathrm{m}}\left(\vec{r}_{0-}\right)$ and $V_{\mathrm{m}}\left(\vec{r}_{0+}\right)$ are the magnetic potentials on the magnet surface in and out of the magnet, we get the final solution of

$$V_{\mathrm{m}}(r, \theta) = \frac{\vec{P}_M \cdot \vec{r}}{4\pi r^3}. \quad (r \geq r_0) \tag{1.67}$$

The above expression being the same with the last expression of (1.45), the expressions of $\vec{H}_{\mathrm{m}}$ (1.46) and $\vec{A}_{\mathrm{m}}$ (1.49) also hold anywhere outside the spherical magnet.

Note 4

$$\nabla \times \vec{B}_I = \mu_0 \nabla \times \vec{H}_I = \nabla \times \nabla \times \vec{A}_I = \nabla\nabla\cdot\vec{A}_I - \nabla^2 \vec{A}_I = -\nabla^2 \vec{A}_I = \mu_0 \vec{j}.$$
$$((1.30), (1.31), (a3.12), (1.14)) \tag{1.68}$$

The last relation of the above equation has the same form with

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{1.69}$$

for the electric potential V in vacuum. Here $\vec{A}_I$, $\vec{j}$, and μ_0 correspond to V, ρ, and $1/\varepsilon_0$, respectively. The solution of (1.69) being well known as

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho \mathrm{d}v}{r}, \tag{1.70}$$

the solution of the last relation of (1.68) is obtained directly as (1.50).

Note 5

$$\mu_0 \int_\infty \vec{H}_I \cdot \vec{H}_{\rm m} {\rm d}v = \int_\infty \left(\nabla \times \vec{A}_I\right) \cdot \vec{H}_{\rm m} {\rm d}v \quad (1.31)$$
$$= \int_\infty \left[\vec{A}_I \cdot \left(\nabla \times \vec{H}_{\rm m}\right) + \nabla \cdot \left(\vec{A}_I \times \vec{H}_{\rm m}\right)\right] {\rm d}v \quad ({\rm a}3.17)$$
$$= \oint_\infty \left(\vec{A}_I \times \vec{H}_{\rm m}\right) \cdot {\rm d}\vec{S} = 0. \quad ((1.32), ({\rm a}3.25),$$
$$\vec{A}_I(r \to \infty) \propto \oint_l \frac{{\rm d}\vec{l}}{r} = \int {\rm d}\vec{S}_{\rm coil} \times \nabla \frac{1}{r} \propto \frac{1}{r^2} \quad ({\rm a}3.24),$$
$$H_{\rm m}(r \to \infty) \propto \frac{1}{r^3} \quad (1.46),\ S(r \to \infty) \propto r^2) \tag{1.71}$$

$$\int_\infty \vec{B}_{\rm m} \cdot \vec{H}_{\rm m} {\rm d}v = \int_\infty \left(\nabla \times \vec{A}_{\rm m}\right) \cdot \vec{H}_{\rm m} {\rm d}v$$
$$= \frac{1}{\mu_0} \int_\infty \left[\vec{A}_{\rm m} \cdot \left(\nabla \times \vec{H}_{\rm m}\right) + \nabla \cdot \left(\vec{A}_{\rm m} \times \vec{H}_{\rm m}\right)\right] {\rm d}v \quad ({\rm a}3.17)$$
$$= \oint_\infty \left(\vec{A}_{\rm m} \times \vec{H}_{\rm m}\right) \cdot {\rm d}\vec{S} = 0. \quad ((1.32), ({\rm a}3.25),$$
$$\vec{A}_{\rm m}\ (\vec{r} \to \infty) \propto = \frac{1}{r^2} \quad (1.49) \tag{1.72}$$

$$\int_\infty \vec{B} \cdot \vec{H}_{\rm m} {\rm d}v = \int_\infty \left(\vec{B}_{\rm m} + \mu_0 \vec{H}_I\right) \cdot \vec{H}_{\rm m} {\rm d}v = 0. \tag{1.73}$$

1.3 Zeeman Energy, Magnetization Energy, and Magnetostatic Energy

1.3.1 Zeeman Energy

Assume that a static magnetic field is distributed in a finite space into which a magnetic moment is brought to position $\vec{r}$ in vacuum from infinite distance. The magnetic polarization moment at the position is $\vec{p}_J\ (\vec{r})$. The work done by the outside force which is the change in the energy of the system consisted of the field and the magnetic moment is

$$E = -\int_{\infty}^{\vec{r}} \frac{\vec{p}_J \cdot \partial \vec{H}}{\partial \vec{r}} \cdot \mathrm{d}\vec{r} = -\int_{0}^{\vec{H}(\vec{r})} \vec{p}_J \cdot \mathrm{d}\vec{H} \qquad (1.16)$$

$$= -\vec{p}_J(\vec{r}) \cdot \vec{H}(\vec{r}) + \int_{\vec{p}_J(\infty)}^{\vec{p}_J(\vec{r})} \vec{H} \cdot \mathrm{d}\vec{p}_J. \qquad (1.74)$$

If $\vec{p}_J(\vec{r})$ remains unchanged during the process, the work done is

$$-\vec{p}_J(\vec{r}) \cdot \vec{H}(\vec{r}), \qquad (1.75)$$

which is the interaction energy between $\vec{p}_J(\vec{r})$ and $\vec{H}(\vec{r})$, that is, the potential energy of the magnetic polarization moment in the field. This kind of potential energy is called Zeeman energy.

1.3.2 Magnetization Energy

The last expression in (1.74) also includes the integral of

$$\vec{H} \cdot \mathrm{d}\vec{p}_J. \qquad (1.76)$$

Apparently $\vec{H} \cdot \mathrm{d}\vec{p}_J$ is the work done to change the magnetic polarization moment by $\mathrm{d}\vec{p}_J$ in the field $\vec{H}$, which is called magnetization energy.

1.3.3 Magnetostatic Energy

1.3.3.1 Magnetic Energy in an Electromagnetic System

Assume that a system of electromagnetic media, electric sources, and steady electric currents are distributed within a finite space. It holds

$$\int_{\infty} \left[\vec{E} \cdot \left(\vec{j} + \frac{\partial \vec{D}}{\partial t} \right) + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right] \mathrm{d}v$$

$$= \int_{\infty} \left[\vec{E} \cdot \left(\nabla \times \vec{H} \right) - \vec{H} \cdot \left(\nabla \times \vec{E} \right) \right] \mathrm{d}v \qquad ((1.9), (1.7))$$

$$= \int_{\infty} \nabla \cdot \left(\vec{H} \times \vec{E} \right) \mathrm{d}v$$

$$= \oint\limits_{S=\infty} \left(\vec{H} \times \vec{E}\right) \cdot \mathrm{d}\vec{S} = 0, \qquad ((\text{a}3.17), (\text{a}3.25)) \tag{1.77}$$

$$\vec{j} = \sigma \left(\vec{E} + \vec{E}_\mathrm{S}\right), \tag{1.78}$$

where σ is electric conductivity and $\vec{E}_\mathrm{S}$ is the equivalent electric field of electromotive force $\oint_l \vec{E}_\mathrm{S} \cdot \mathrm{d}\vec{l}$ of non-electric origin. The energy $\int\int \vec{j} \cdot \vec{E}_\mathrm{S} \mathrm{d}v \mathrm{d}t$ provided by the electric source transforms to three kinds of energy as

$$\int\int \vec{j} \cdot \vec{E}_\mathrm{S} \mathrm{d}v \mathrm{d}t = \int\int \vec{j} \cdot \left(\frac{\vec{j}}{\sigma} - \vec{E}\right) \mathrm{d}v \mathrm{d}t \qquad (1.78)$$

$$= \int\int \frac{j^2}{\sigma} \mathrm{d}v \mathrm{d}t + \int\int \vec{E} \cdot \mathrm{d}\vec{D} \mathrm{d}v + \int\int \vec{H} \cdot \mathrm{d}\vec{B} \mathrm{d}v. \qquad (1.77) \tag{1.79}$$

The first, second, and third terms in the last expression are the Joule heat, the change in electric energy, and magnetic energy.

1.3.3.2 Magnetic Field Energy

The magnetic energy $\int\int \vec{H} \cdot \mathrm{d}\vec{B} \mathrm{d}v$ can be broken down into two parts:

$$\int\int \vec{H} \cdot \mathrm{d}\vec{B} \mathrm{d}v = \int\int \vec{H} \cdot \mathrm{d}\vec{J} \mathrm{d}v + \int \frac{\mu_0 H^2}{2} \mathrm{d}v. \tag{1.80}$$

The first term in the right is the magnetization energy. The second term is the magnetic field energy which is the sum of the magnetic field energy of the field produced by currents and that produced by magnetic media:

$$\int \frac{\mu_0 H^2}{2} \mathrm{d}v = \int \frac{\mu_0 H_I^2}{2} \mathrm{d}v + \int \frac{\mu_0 H_\mathrm{m}^2}{2} \mathrm{d}v. \qquad ((1.29), (1.54)) \tag{1.81}$$

1.3.3.3 Magnetic Moment Interaction Energy, Magnetic Charge Interaction Energy, and Magnetostatic Energy

The magnetic field energy produced by magnetic moments can be reformulated as

$$\int \frac{\mu_0 H_\mathrm{m}^2}{2} \mathrm{d}v = -\frac{1}{2} \int \vec{J} \cdot \vec{H}_\mathrm{m} \mathrm{d}v = \frac{1}{2} \left(\int \rho_\mathrm{m} V_\mathrm{m} \mathrm{d}v + \oint \sigma_\mathrm{m} V_\mathrm{m} \mathrm{d}S\right).$$
(Note 1 of this section) (1.82)

The second expression of the above equation represents the interaction energy between magnetic moments. The third expression represents the interaction energy between magnetic charges. In brief summary, all of the field energy of the field produced by magnetic moments, the interaction energy between magnetic moments, and the interaction energy between magnetic charges refer to the same energy. The only differences between them are the physical pictures, and they are called magnetostatic energy.

Note 1

Since

$$\int \vec{B}\cdot\vec{H}_{\rm m}{\rm d}v = \int \left[\mu_0\left(\vec{H}_I + \vec{H}_{\rm m}\right) + \vec{J}\right]\cdot\vec{H}_{\rm m}{\rm d}v = 0, \qquad (1.54) \qquad (1.83)$$

it holds

$$\begin{aligned}\int \frac{\mu_0 H_{\rm m}^2}{2}{\rm d}v &= -\frac{1}{2}\int \vec{J}\cdot\vec{H}_{\rm m}{\rm d}v \qquad (1.54)\\ &= \frac{1}{2}\int \vec{J}\cdot(\nabla V_{\rm m}){\rm d}v = \frac{1}{2}\int\left[-(\nabla\cdot\vec{J})V_{\rm m} + \nabla\cdot\left(\vec{J}V_{\rm m}\right)\right]{\rm d}v\\ &\quad ((1.34), (a3.16))\\ &= \frac{1}{2}\left(\int \rho_{\rm m}V_{\rm m}{\rm d}v + \oint \sigma_{\rm m}V_{\rm m}{\rm d}S\right).\\ &\quad ((1.27), (a3.25), (1.39))\end{aligned} \qquad (1.84)$$

1.4 Thermodynamics for Magnetic Media

1.4.1 Principles of Thermodynamics for Magnetic Media

The increase in internal energy of a system $\mathrm{d}U$ equals to the sum of the heat supplied to the system δQ and the work done on the system $\sum_i X_i \mathrm{d}x_i$:

$$\mathrm{d}U = \delta Q + \sum_i X_i \mathrm{d}x_i. \qquad (1.85)$$

Here X_i and x_i are a generalized force and coordinate,

$$\delta Q \le T\mathrm{d}S, \qquad (1.86)$$

where T and S are the temperature and entropy. The equal and non-equal relation of (1.86) applies to reversible and irreversible processes, respectively.

The entropy of an isolated system increases steadily during an irreversible process and reaches a maximum when the system stabilizes.

In the following a unit volume of an infinite magnetic medium of uniform magnetization under the atmospheric pressure is considered as the system in order to exclude magnetostatic energy from consideration. During a reversible process, the change of the internal energy is

$$\begin{aligned} \mathrm{d}U &= T\mathrm{d}S + \sum_i X_i \mathrm{d}x_i = T\mathrm{d}S + \vec{H}\cdot\mathrm{d}\vec{J} + \sum_i^{X_i\mathrm{d}x_i \neq \vec{H}\cdot\mathrm{d}\vec{J}} X_i \mathrm{d}x_i \\ &= T\mathrm{d}S + H\mathrm{d}J + \sum_i^{X_i\mathrm{d}x_i \neq H\mathrm{d}J} X_i \mathrm{d}x_i, \end{aligned} \tag{1.87}$$

where $\sum_i^{X_i\mathrm{d}x_i \neq H\mathrm{d}J} X_i\mathrm{d}x_i$ is the work other than $H\mathrm{d}J$.

1.4.2 Free Energy and Thermal Potential

The free energy (Helmholtz free energy) F, thermal potential (Gibbs's free energy) G, and their differentials are

$$F \equiv U - TS, \tag{1.88}$$

$$\mathrm{d}F = -S\mathrm{d}T + \vec{H}\cdot\mathrm{d}\vec{J} + \sum_i^{X_i\mathrm{d}x_i \neq \vec{H}\cdot\mathrm{d}\vec{J}} X_i\mathrm{d}x_i = -S\mathrm{d}T + H\mathrm{d}J + \sum_i^{X_i\mathrm{d}x_i \neq H\mathrm{d}J} X_i\mathrm{d}x_i, \tag{1.89}$$

$$G \equiv F - \sum_i X_i x_i = F - \vec{H}\cdot\vec{J} - \sum_i^{X_i x_i \neq \vec{H}\cdot\vec{J}} X_i x_i = F - HJ - \sum_i^{X_i x_i \neq HJ} X_i x_i, \tag{1.90}$$

$$\begin{aligned} \mathrm{d}G &= -S\mathrm{d}T - \sum_i x_i \mathrm{d}X_i = -S\mathrm{d}T - \vec{J}\cdot\mathrm{d}\vec{H} - \sum_i^{x_i\mathrm{d}X_i \neq \vec{J}\cdot\mathrm{d}\vec{H}} x_i\mathrm{d}X_i \\ &= -S\mathrm{d}T - J\mathrm{d}H - \sum_i^{x_i\mathrm{d}X_i \neq J\mathrm{d}H} x_i\mathrm{d}X_i. \end{aligned} \tag{1.91}$$

$-\vec{H}\cdot\vec{J} = -HJ$ in (1.90) is the potential energy of the magnetization in the field, i.e., the interaction energy between the system and the field applied from outside. If the system is selected to include the field, $-\vec{H}\cdot\vec{J} = -HJ$ becomes a part of the energy of the system and G of (1.90) is no longer the thermal potential but becomes the free energy of the system.

1.4.3 Stabilization Conditions for a System at Constant Temperature

Suppose X_i and x_i other than H and J remain constant. The stabilization conditions for a system at constant temperature are:

In the absence of magnetic field, $\vec{J}$ takes the value and direction to make the F minimum.

In the presence of magnetic field, $\vec{J}$ takes the value and direction to make the G minimum (Note 1 of this section).

With the above characteristics of F and G, they are often tersely called energy.

It can be deduced from $\mathrm{d}F = 0$ (1.89) and $\mathrm{d}G = 0$ (1.91) that at an equilibrium state under a constant temperature the following relations hold:

$$\vec{H} - \frac{\partial F}{\partial \vec{J}} = 0, \qquad H - \frac{\partial F}{\partial J} = 0. \tag{1.92}$$

$$\vec{J} = -\frac{\partial G}{\partial \vec{H}}, \qquad J = -\frac{\partial G}{\partial H}. \tag{1.93}$$

The relations of (1.92) are the equilibrium conditions of the forces that the sum of the force acting on the system $\vec{H}$ (or H) and internal force $-\frac{\partial F}{\partial \vec{J}}$ (or $-\frac{\partial F}{\partial J}$) equals zero. Equation (1.93) shows that $\vec{J}$ (or J) equals to $-\frac{\partial G}{\partial \vec{H}}$ (or $-\frac{\partial G}{\partial H}$). For a system consisting of N same magnetic ions, G is a function of the distribution function Z:

$$G = -k_{\mathrm{B}} T N \ln Z. \qquad (k_{\mathrm{B}}\text{: Boltzmann constant}) \tag{1.94}$$

The localized magnetic ions being distinguishable the classic statistics is applicable and Z is

$$Z = \sum_n \exp\left(-\frac{E_n}{k_{\mathrm{B}} T}\right), \tag{1.95}$$

where E_n is the n^{th} eigenenergy of the ion. If $\{E_n\}$ is known, J can be calculated from the relation of

$$J = -\frac{\partial G}{\partial H} = -\frac{N}{Z} \sum_n \frac{\partial E_n}{\partial H} \exp\left(-\frac{E_n}{k_{\mathrm{B}} T}\right). \tag{1.96}$$

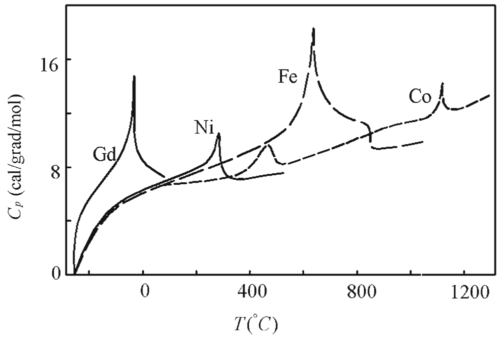


Fig. 1.6 Specific heat vs temperature for Fe, Co, Ni, and Gd metals [3]

1.4.4 First- and Second-Order Transformations

First consider the crystal structure transformation. The thermal potential of the crystal and its differential are

$$G = F + pv - \sum_i^{X_i x_i \neq -pv} X_i x_i, \tag{1.97}$$

$$\mathrm{d}G = v\mathrm{d}p - S\mathrm{d}T, \quad \left(\sum_i^{x_i \mathrm{d}X_i \neq -v\mathrm{d}p} x_i \mathrm{d}X_i = 0 \right) \tag{1.98}$$

where p is the pressure in the system and v is the volume of the crystal. The transformation accompanies discontinuous change of the volume $v = \partial G/\partial p$ and absorption or release of the latent heat caused by rearrangement of the atoms. The latent heat is the product of the change of the entropy and the phase transformation temperature which remains constant. Therefore, $S = -\partial G/\partial T$ also changes discontinuously. Such a transformation for which the first partial derivatives of G with respect to temperature and general force change discontinuously with temperature is called first-order transformation. This kind of transformation has some hysteresis phenomenon, i.e., the phase transformation temperatures are somewhat different for the opposite phase transformations.

Magnetic transformation, such as ferromagnetism$\leftrightarrow$ paramagnetism, is different from the first-order transformation. The first-order partial derivatives of G with respect to temperature and general force, $S = -\partial G/\partial T$ and $J = -\partial G/\partial H$, change continuously. But the specific heat $C_p = T\partial S/\partial T = -T\partial^2 G/\partial T^2$ and differential susceptibility $\partial J/\partial H = -\partial^2 G/\partial H^2$ which are the second-order partial derivatives change discontinuously (Figs. 1.6 and 1.7). Such kind of transformation is called second-order transformation.

A small number of magnetic media abruptly and markedly change J through crystal structure or magnetic transformation, such as from monoclinic to orthorhombic

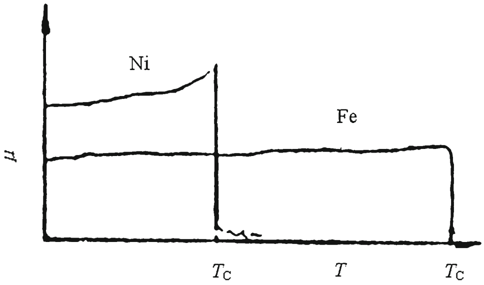


Fig. 1.7 AC permeability (0.16 kA/m, 5 kHz) vs temperature for Fe and Ni metals [4]

or from ferrimagnetic to ferromagnetic transformation, when H is changed. Such kind of transition induced by magnetic field is called metamagnetic transition.

1.4.5 Magneto-Caloric Effect

Let

$$U' = U - HJ. \tag{1.99}$$

During a reversible process

$$\mathrm{d}U' = T\mathrm{d}S - J\mathrm{d}H. \tag{1.100}$$

Take T and H as the independent variables. During an adiabatic process dS is zero:

$$\mathrm{d}S = \frac{1}{T}\left[\frac{\partial U'}{\partial T}\mathrm{d}T + \left(\frac{\partial U'}{\partial H} + J\right)\mathrm{d}H\right] = 0, \qquad (1.100) \tag{1.101}$$

where

$$\frac{\partial U'}{\partial T} = T\frac{\partial S}{\partial T} = C_H \tag{1.102}$$

is the specific heat in a constant magnetic field and

$$\frac{\partial U'}{\partial H} + J = T\frac{\partial S}{\partial H} = -T\frac{\partial}{\partial H}\frac{\partial G}{\partial T} = T\frac{\partial J}{\partial T}. \tag{1.103}$$

Inserting (1.102) and (1.103) into the second relation of (1.101) and then integrating the latter we get the change of temperature ΔT_{ad} induced by application of magnetic field as,

$$\Delta T_{\mathrm{ad}} = -\int_0^H \frac{T}{C_H}\frac{\partial J}{\partial T}\mathrm{d}H. \tag{1.104}$$

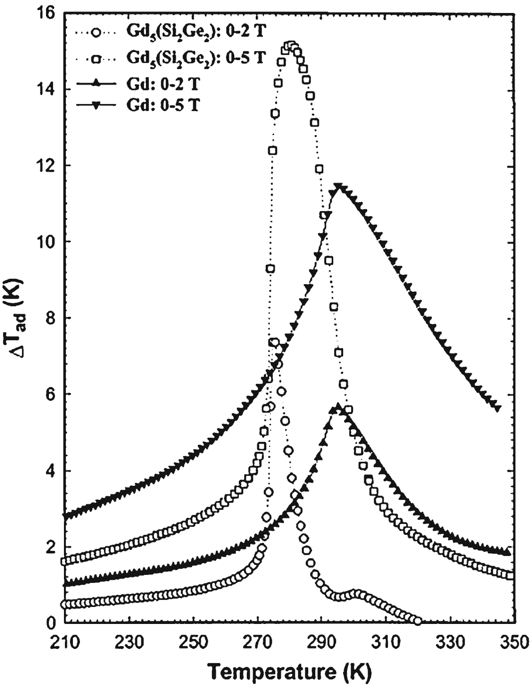


Fig. 1.8 ΔT_{ad} vs T for Gd metal (T_C=293 K) and $Gd_5Si_2Ge_2$($T_C = 276 \sim 299K$). $\mu_0 H$ is varied between 0 and 2 T and 0 and 5 T [5]

This phenomenon is called magneto-caloric effect. The effect has been widely applied in the cooling at low temperatures.

In recent years the exploration of magneto-caloric materials for cooling around R.T. is active. These materials should be able to produce large ΔT_{ad} at around R.T. by applying a not very large magnetic field. To satisfy these conditions, the material should have a large value of $\partial J/\partial T$, i.e., a large J_s and Curie temperature near R. T.. Gd metal satisfies these conditions relatively well. A large ΔT_{ad} can also be obtained by use of metamagnetic materials with metamagnetic transition temperature near R.T.. $Gd_5Si_2Ge_2$ belongs to such kind of materials. Figure 1.8 demonstrates the ΔT_{ad} as a function of temperature for Gd and $Gd_5Si_2Ge_2$ under the variation of H.

Note 1

Let δS and δS_0 denote the change of entropy of the magnetic medium and the infinitely large isothermal heat source, respectively, after an irreversible process. According to the second law of thermodynamics, the total entropy increases:

$$\delta S + \delta S_0 > 0. \tag{1.105}$$

Under the condition of constant temperature, the exchange of heat is reversible for the infinite isothermal heat source, so

$$\delta S_0 = -\frac{\delta Q}{T}. \tag{1.106}$$

When $H = 0$, the heat transferred to the medium is $\delta Q = \delta U$ and hence $\delta S_0 = -\delta U/T$. Inserting this relation into (1.105), we get

$$T\delta S - \delta U = -\delta F > 0, \tag{1.107}$$

that is, F decreases until it reaches a minimum.

Under the conditions of constant temperature and applied magnetic field,

$$\delta S_0 = -\frac{\delta Q}{T} = -\frac{\delta(U - HJ)}{T}. \tag{1.108}$$

Inserting the above relation into (1.105), we get

$$T\delta S - \delta(U - HJ) = -\delta(U - TS - HJ) = -\delta G > 0, \tag{1.109}$$

that is, G decreases until it reaches a minimum.

1.5 Hamiltonian of an Electric Charged Particle in Static Electric and Magnetic Fields

Let m and e denote the mass and electric charge of the particle.

1.5.1 Hamiltonian and Momentum in Classical Mechanics

In the classical mechanics the Hamiltonian of a particle is defined by

$$H(\vec{q}, \vec{p}; t) \equiv \left[\dot{\vec{q}}\cdot\vec{p} - L\left(\vec{q}, \dot{\vec{q}}; t\right)\right]_{\dot{\vec{q}}\to\vec{p}}, \qquad \left(\dot{\vec{q}} \equiv \frac{d\vec{q}}{dt}, t : \text{time}\right) \tag{1.110}$$

where $\vec{q}$ is the generalized coordinate and $\vec{p}$ is the generalized momentum conjugates to the coordinate $\vec{q}$ defined by

$$\vec{p} \equiv \frac{\partial L\left(\vec{q}, \dot{\vec{q}}; t\right)}{\partial\dot{\vec{q}}}. \tag{1.111}$$

Here L is the Lagrangian which satisfies the Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\left(\vec{q},\dot{\vec{q}};t\right)}{\partial\dot{\vec{q}}}-\frac{\partial L\left(\vec{q},\dot{\vec{q}};t\right)}{\partial\vec{q}}=0. \tag{1.112}$$

The $\vec{p}$, L, and H in static electric and magnetic fields are

$$\vec{p}=m\dot{\vec{q}}+e\vec{A}\left(\vec{q}\right), \qquad \left(\vec{A}\text{: magnetic vector potential}\right) \tag{1.113}$$

$$L\left(\vec{q},\dot{\vec{q}};t\right)=\frac{1}{2}m\dot{q}^2-e\left[V\left(\vec{q}\right)-\dot{\vec{q}}\cdot\vec{A}\left(\vec{q}\right)\right], \tag{1.114}$$

$$H\left(\vec{q},\vec{p};t\right)=\frac{1}{2m}\left[\vec{p}-e\vec{A}\left(\vec{q}\right)\right]^2+eV\left(\vec{q}\right)=\frac{(m\dot{q})^2}{2m}+eV. \tag{1.115}$$

Above relations show that $\vec{p}$ is the usual momentum $m\dot{\vec{q}}$ in the absence of magnetic field but is not in a magnetic field, and the Hamiltonian is the total energy irrelevant to the magnetic field.

1.5.2 Hamiltonian and Momentum in Quantum Mechanics

In quantum mechanics the Hamiltonian is an operator $\hat{H}$ which has the same form of (1.115) of classical mechanics and also represents the total energy. The operator of generalized momentum $\vec{p}$ is

$$\hat{\vec{p}}=-\mathrm{i}\hbar\nabla. \qquad (\hbar\text{: reduced Planck constant}) \tag{1.116}$$

Here $\hat{\vec{p}}$ rather than $m\dot{\vec{q}}$ is considered the momentum. Thus, in contrast to the Hamiltonian of classical mechanics (1.115), it includes not only the kinetic energy $\hat{\vec{p}}^2/2m$ and electric potential energy eV but also the energy related to the magnetic field.

Corresponding to $\vec{p}_J=-\partial E/\partial\vec{H}$ (1.18) of classical mechanics, the operator of magnetic polarization moment in quantum mechanics is

$$\hat{\vec{p}}_J=-\frac{\partial\hat{H}}{\partial\vec{H}}. \tag{1.117}$$

In the classical mechanics the total energy being irrelevant to magnetic field, derivative of the energy with respect to magnetic field is zero, and the magnetic moment is independent of magnetic field and is always zero. This result contradicts with reality, which reveals that the classical mechanics cannot explain magnetic phenomena of media. This fact is called Bohr-van Leeuwen theorem [6].

In quantum mechanics $\hat{H}$ being related with magnetic field, the magnetic moment may not be zero. In fact, magnetism of media is quantum phenomena.

Introducing some results of quantum mechanics, such as a magnetic atom has an intrinsic magnetic moment and there exists exchange interactions between electron spins, into the frame of classical mechanics, however, many magnetic phenomena can be explained successfully with convenience. Such method is used widely.

Appendix 1: Physical constants

Table a1.1 Physical constants [7]

Physical quantity		SI unit	CGS unit
Avogadro constant	N_A	$6.022{,}142\times10^{23}$ mol^{-1}	$6.022{,}142\times10^{23}$ mol^{-1}
Bohr magneton	$\mu_B = \mu_0\lvert e\rvert\hbar/(2m)$ $= \lvert e\rvert\hbar/(2m)$	$1.165{,}406\times10^{-29}$ J · m/A $9.274{,}009\times10^{-24}$ J/T	$9.274{,}009\times10^{-21}$ erg/Oe
Boltzmann constant	k_B	$1.380{,}649\times10^{-23}$ J/K	$1.380{,}649\times10^{-16}$ erg/K
Electron charge	e	$-1.602{,}176{,}6\times10^{-19}$ C	$-4.803{,}204\times10^{-10}$ esu
Electron rest mass	m	$9.109{,}382\times10^{-31}$ kg	$9.109{,}382\times10^{-28}$ g
Reduced Planck constant	$\hbar$	$1.054{,}571{,}7\times10^{-34}$ J · s	$1.054{,}571{,}7\times10^{-27}$ erg · s
Magnetic constant	μ_0	$4\pi\times10^{-7}$ N/A^2	1
Speed of light in vacuum	c	$2.997{,}924{,}58\times10^{8}$ m/s	$2.997{,}924{,}58\times10^{10}$ cm/s
Electric constant	$\varepsilon_0 = 1/(\mu_0 c^2)$	$8.854{,}187{,}817\times10^{-12}$ F/m	1

Appendix 2: Units and Their Conversions

Table a2.1 Formulae

SI	CGS
$\vec{B} = \mu_0\left(\vec{H}+\vec{M}\right), \quad \vec{J} = \mu_0\vec{M}$	$\vec{B} = \vec{H} + 4\pi\vec{M}$
$f = \dfrac{q_{m1}q_{m2}}{4\pi\mu_0 r^2}$	$f = \dfrac{q_{m1}q_{m2}}{r^2}$
$\int \vec{H}\cdot d\vec{B}$	$\dfrac{1}{4\pi}\int \vec{H}\cdot d\vec{B}$
$\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	$\nabla\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}$
$\nabla\times\vec{H} = \vec{j} + \dfrac{\partial\vec{D}}{\partial t}$	$\nabla\times\vec{H} = \dfrac{4\pi}{c}\vec{j} + \dfrac{1}{c}\dfrac{\partial\vec{D}}{\partial t}$
$\nabla\cdot\vec{B} = 0$	$\nabla\cdot\vec{B} = 0$
$\nabla\cdot\vec{D} = \rho$	$\nabla\cdot\vec{D} = 4\pi\rho$

Table a2.2 Units and Their Conversions

Magnetic quantity	SI		CGS		Unit conversion
Magnetic field	H	A/m	H	Oe	$1\text{Oe} = 10^3/4\pi\,\text{A/m}$
Magnetic induction	B	$\text{T} = \text{Wb/m}^2$	B	Gs	$1\text{Gs} = 10^{-4}\text{T}$
Magnetic polarization moment	p_J	Wb·m		emu	$1\text{emu} = 4\pi \times 10^{-10}\text{Wb·m}$
Magnetic moment	p_M	A·m^2		emu	$1\text{emu} = 1 \times 10^{-3}\text{A·m}^2$
Magnetization	M	A/m	M	Gs	$1\text{Gs} = 10^3\,\text{A/m}$
Magnetic polarization	J	T	$4\pi M$	Gs	$1\text{Gs} = 10^{-4}\text{T}$
Magnetic energy product	$(BH)_{\max}$	J/m^3	$(BH)_{\max}$	Gs·Oe	$1\text{Gs·Oe} = 0.1/4\pi\ \text{J/m}^3$
Magnetic flux	ϕ	Wb	ϕ	Mx	$1\text{Mx} = 10^{-8}\,\text{Wb}$
Magnetic constant	μ_0	$\text{N/A}^2 = \text{H/m}$			

Appendix 3: Selections From Vector Analysis

Vector Algebra

$$\vec{a}\cdot\vec{b} = \sum_{\alpha}^{x,y,z} a_\alpha b_\alpha. \tag{a3.1}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left(a_y b_z - a_z b_y\right)\vec{e}_x + \left(a_z b_x - a_x b_z\right)\vec{e}_y + \left(a_x b_y - a_y b_x\right)\vec{e}_z. \tag{a3.2}$$

($\vec{e}_\alpha$: unit vector in the α direction)

$$\vec{a}\cdot\left(\vec{b} \times \vec{c}\right) = \vec{b}\cdot\left(\vec{c} \times \vec{a}\right) = \vec{c}\cdot\left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{a3.3}$$

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\left(\vec{a}\cdot\vec{c}\right) - \vec{c}\left(\vec{a}\cdot\vec{b}\right). \tag{a3.4}$$

Del, Gradient, Divergence, and Curl Operators

$$\nabla \equiv \sum_{\alpha}^{x,y,z} \vec{e}_\alpha \frac{\partial}{\partial \alpha}. \qquad \text{(Del or Nabla operator)} \tag{a3.5}$$

$$\nabla \cdot \nabla = \nabla^2 = \sum_{\alpha}^{x,y,z} \frac{\partial^2}{\partial \alpha^2}. \qquad \text{(Laplace operator)} \tag{a3.6}$$

$$\nabla \varphi \equiv \left(\frac{d\varphi}{d\vec{l}}\right)_{\max} = \sum_{\alpha}^{x,y,z} \frac{\partial \varphi}{\partial \alpha} \vec{e}_\alpha. \qquad \text{(gradient of } \varphi\text{)} \tag{a3.7}$$

$$\nabla \cdot \vec{a} \equiv \lim_{v \to 0} \frac{\oint \vec{a} \cdot d\vec{S}}{v} = \sum_{\alpha}^{x,y,z} \frac{\partial a_\alpha}{\partial \alpha}. \qquad \text{(divergence of } \vec{a}, v: \text{ volume surrounded by the closed surface S)} \tag{a3.8}$$

$$\nabla \times \vec{a} \equiv \lim_{S \to 0} \left(\frac{\oint \vec{a} \cdot \mathrm{d}\vec{l}}{\vec{S}}\right)_{\max} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}. \qquad \text{(curl of } \vec{a}, S\text{: area of the closed curve } l\text{)} \tag{a3.9}$$

$$\nabla \times \nabla \varphi = 0. \tag{a3.10}$$

$$\nabla \cdot \nabla \times \vec{a} = 0. \tag{a3.11}$$

$$\nabla \times \nabla \times \vec{a} = \left(\nabla \nabla \cdot - \nabla^2\right) \vec{a}. \tag{a3.12}$$

If $\nabla \times \vec{a} = 0$ in region D, $\vec{a}$ is a divergence of a scalar in D:

$$\vec{a} = \nabla \varphi. \tag{a3.13}$$

If $\nabla \cdot \vec{a} = 0$ in region D, $\vec{a}$ is a curl of a vector in D:

$$\vec{a} = \nabla \times \vec{A}. \tag{a3.14}$$

Del Operations on Products of Two Functions

$$\nabla\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \times \left(\nabla \times \vec{b}\right) + \vec{b} \times \left(\nabla \times \vec{a}\right) + \left(\vec{a} \cdot \nabla\right)\vec{b} + \left(\vec{b} \cdot \nabla\right)\vec{a}. \tag{a3.15}$$

$$\nabla \cdot \left(\varphi \vec{a}\right) = \left(\nabla \varphi\right) \cdot \vec{a} + \varphi \nabla \cdot \vec{a}. \tag{a3.16}$$

$$\nabla\cdot\left(\vec{a}\times\vec{b}\right)=\left(\nabla\times\vec{a}\right)\cdot\vec{b}-\vec{a}\cdot\left(\nabla\times\vec{b}\right). \tag{a3.17}$$

$$\nabla\times\left(\varphi\vec{a}\right)=\left(\nabla\varphi\right)\times\vec{a}+\varphi\nabla\times\vec{a}. \tag{a3.18}$$

$$\nabla\times\left(\vec{a}\times\vec{b}\right)=\left(\vec{b}\cdot\nabla\right)\vec{a}-\left(\vec{a}\cdot\nabla\right)\vec{b}+\left(\nabla\cdot\vec{b}\right)\vec{a}-\left(\nabla\cdot\vec{a}\right)\vec{b}. \tag{a3.19}$$

Del Operations in Spherical Coordinate System

$$\nabla\varphi=\frac{\partial\varphi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi. \tag{a3.20}$$

$$\nabla\cdot\vec{a}=\frac{1}{r^2}\frac{\partial\left(a_r r^2\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(a_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}. \tag{a3.21}$$

$$\begin{aligned}\nabla\times\vec{a}=&\frac{1}{r\sin\theta}\left[\frac{\partial\left(a_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial a_\theta}{\partial\phi}\right]\vec{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi}-\frac{\partial\left(a_\phi r\right)}{\partial r}\right]\vec{e}_\theta\\&+\frac{1}{r}\left[\frac{\partial\left(a_\theta r\right)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_\phi.\end{aligned} \tag{a3.22}$$

$$\nabla^2\varphi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{\left(r\sin\theta\right)^2}\frac{\partial^2\varphi}{\partial\phi^2}. \tag{a3.23}$$

Integral Relations

$$\int \mathrm{d}\vec{S}\times\nabla\varphi=\oint\varphi\mathrm{d}\vec{l}. \tag{a3.24}$$

$$\int \nabla\cdot\vec{a}\mathrm{d}v=\oint\vec{a}\cdot\mathrm{d}\vec{S}. \tag{a3.25}$$

$$\int \nabla\times\vec{a}\mathrm{d}v=\oint\mathrm{d}\vec{S}\times\vec{a}. \tag{a3.26}$$

$$\int \left(\nabla\times\vec{a}\right)\cdot\mathrm{d}\vec{S}=\oint\vec{a}\cdot\mathrm{d}\vec{l}. \tag{a3.27}$$

$$\int\left[\psi\nabla^2\varphi+(\nabla\psi)\cdot(\nabla\varphi)\right]\mathrm{d}v=\oint\psi(\nabla\varphi)\cdot\mathrm{d}\vec{S}. \tag{a3.28}$$

$$\int\left(\psi\nabla^2\varphi-\varphi\nabla^2\psi\right)\mathrm{d}v=\oint\left(\psi\nabla\varphi-\varphi\nabla\psi\right)\cdot\mathrm{d}\vec{S}. \tag{a3.29}$$

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Chapter 2
Magnetism of Atoms

The magnetic moments of magnetic materials originate from magnetic moments of the atoms. The hyperfine interactions have important applications in magnetism. This chapter introduces the basic theories relating to atomic magnetism and hyperfine interactions in atom. It contains the sections of Electron configuration of atom; Term and multiplet; Intrinsic magnetic moment and gyromagnetic ratio of atom; Paramagnetism and diamagnetism of atom; Exchange interaction in He atom; Exchange interaction in H_2 molecule; and Hyperfine interactions in atom; and Appendices 4 to 6. The eigenfunctions are assumed orthonormalized.

2.1 Electron Configuration of Atom

Intrinsic atomic magnetic moment is decided by the state of the electrons. Consider an atom or an ion of atomic number Z which has N electrons.

2.1.1 Electron Spin

An electron has an intrinsic angular momentum called spin angular momentum, or tersely spin. The operators of its projections on the x, y and z axes (z axis = quantum axis) are

$$\hat{s}_\alpha \equiv \frac{\hbar}{2}\hat{\sigma}_\alpha, \quad (\alpha = x, y, z), \tag{2.1}$$

where $\hat{\sigma}_\alpha$ is the Pauli spin matrices,

$$\hat{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_\alpha^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

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